A short visit inside Algebraic Combinatorics

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- The Ultimate Goal: provide constructions or proofs requiring (almost) no mathematical knowledge but offering great insights in the theory at work.
- Our enemies: theories with no examples (algebraic nonsense) and the induction process.

Examples

Representation theory!

- Integer partitions encoding the irreducible representations of the symmetric group,
- Standard Young tableaux giving the size of their irreducible representation,
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Today: Operads!

Cut the deck

Start with a deck of card. Cut it in half and shuffle together both subdecks. What happens?

- With 52 cards and two decks of say 26 cards, we get $\binom{52}{26}$ different possibilities.
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Oh sorry!
I'm doing algebraic combinatorics not asymptotics. Too bad, the question is so nice...

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So this operation is commutative and associative!

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Consider two words

$$u = u_1 \dots u_n \qquad v = v_1 \dots v_p$$

Their shuffle $u \sqcup\!\sqcup v$ is

$$u \sqcup\!\sqcup v := (u_1 \dots u_{n-1} \sqcup\!\sqcup v).u_n + (u \sqcup\!\sqcup v_1 \dots v_{p-1}).v_p.$$

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This equation is clearly a sum of two parts. Separate these parts.

$$\begin{cases} u < v := (u_1 \dots u_{n-1} \sqcup\!\sqcup v).u_n \\ u > v := (u \sqcup\!\sqcup v_1 \dots v_{p-1}).v_p \end{cases}$$

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Please welcome the *dendriform* operators!

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With three words, there are 8 expressions using $<$ and $>$:

$$\begin{cases} (u < v) < w & u < (v < w) \\ (u < v) > w & u < (v > w) \\ (u > v) < w & u > (v < w) \\ (u > v) > w & u > (v > w) \end{cases}$$

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Do they have other relations?

Right, right

In all expressions

$$\begin{cases} (u < v) < w \\ (u < v) > w \\ (u > v) < w \\ (u > v) > w \end{cases} \qquad \begin{matrix} u < (v < w) \\ u < (v > w) \\ u > (v < w) \\ u > (v > w) \end{matrix}$$

Right, right

In all expressions the last letter comes from a given word:

$$\begin{cases} (u < v) < w \Leftarrow u \\ (u < v) > w \Leftarrow w \\ (u > v) < w \Leftarrow v \\ (u > v) > w \Leftarrow w \end{cases} \qquad \begin{array}{l} u < (v < w) \Leftarrow u \\ u < (v > w) \Leftarrow u \\ u > (v < w) \Leftarrow v \\ u > (v > w) \Leftarrow w \end{array}$$

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Hence

$$\begin{cases} (u < v) < w = u < (v < w) + u < (v > w) \\ (u > v) < w = u > (v < w) \\ (u > v) < w + (u > v) > w = u > (v > w) \end{cases}$$

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And there cannot be other relations with 3 words.

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First, write any dendriform expression as a binary tree:

```
      <
     / \
    >   w
   / \
  u   v
```

$$= (u > v) < w.$$

Forbidden rights

Our relations can be written as rewriting rules on trees:

$$\left\{\begin{array}{l}
\begin{array}{cc} & < \\ / & \\ < & \end{array} \;\to\; \begin{array}{cc} < & \\ & \backslash \\ & < \end{array} + \begin{array}{cc} < & \\ & \backslash \\ & > \end{array} \\[2em]
\begin{array}{cc} > & \\ & \backslash \\ & < \end{array} \;\to\; \begin{array}{cc} & < \\ / & \\ > & \end{array} \\[2em]
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\end{array}\right.$$

Left overs

How many non-rewritable trees are there?

Split them according to their root:

$$\begin{cases} S_< = xS(S - S_<) \\ S_> = xS. \end{cases}$$

so that

$$S = 1 + 2xS + x^2S^2 = (1 + xS)^2.$$

And one easily finds that S is the g.s. of the Catalan numbers.

What's right and what's left (to be done)

All dendriform expressions with n operands can be rewritten as Catalan different (non-rewritable) trees. So the dendriform *operad* on 1 generator (all leaves of the trees equal to 1) has graded dimension at most Catalan. Converse property?

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With the help of combinatorics!

Different rights

Consider the five different trees with $n = 3$:

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When applied to 1 on each leaf, one gets

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Note that these are disjoint sets!

Characterize these sets

Loday proved that two permutations are in the same subset iff their inverses satisfy that their decreasing trees have same shape.

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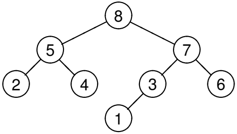
Yes!

And No...

Decreasing trees

If $\sigma = 25481376$, its decreasing tree is



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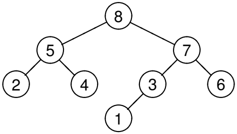

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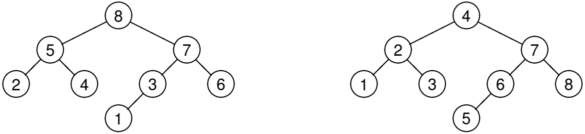

The tree on the right is the *Binary Search tree* of $\sigma^{-1} = 51632874$.

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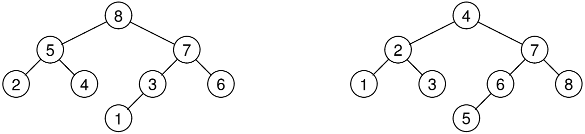

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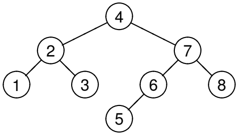
So Loday's result is equivalent to: two permutations have the same image iff they have the same BST.

From BSTs to combinatorics on words

Can someone guess (without computations) other permutations having this same tree as BST?



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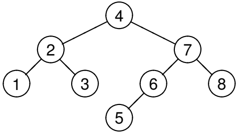

Hint: the extremal ones are 13256874 and 85673124.

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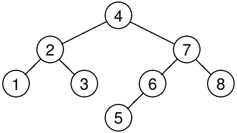

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Complete answer: they are the *linear extensions* of the tree and an *interval* of the weak order on permutations.

A monoid on trees, a sylvester monoid

Given a permutation, finding all permutations with the same BST does not require building the BST itself!
It amounts to compute the transitive closure of the following rewriting rules:

$$ac\ldots b \equiv ca\ldots b \quad \text{for all} \quad a < b < c.$$

This is the *sylvester monoid*.

From monoids to operads

On these objects, a one-line proof shows that $<$ and $>$ of two sylvester classes is a union of sylvester classes.

The converse is also easy to prove: any sylvester class can be obtained as a linear combination of the dendriform operad generated by 1. Write the dendriform expression of their corresponding tree.

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So the free dendriform operad has dimension Catalan exactly.

And so is our instance on permutations which is btw free too.

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- Easier way of designing generalizations: all combinatorial objects have analogs of their own:
  - permutations: packed words: $i \in w \to i-1 \in w$, parking functions, signed permutations, ...
  - binary trees: Cayley trees, Cambrian trees, ...
  - BST and Decreasing trees: repeated letters, fixed number of repeated letters, ...
  - sylvester monoid: plactic, hyposylvester, metasylvester, ...

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- Hook formulas and (q, t)-hooks now available without efforts,
- Noncommutative setting where algebraic proofs come easily, multistatistics on permutations for free, ...

Open problems

Combinatorial questions:

- Study more examples,
- Fill in the blanks: describe combinatorially and enumerate the intervals of orders on permutations, packed words, parking functions, ...

Algebraic or geometrical questions:

- Provide a general setting where the combinatorial algebras are related to polytopes,
- Get a non semi-simple algebra whose representation theory rings encode the (commutative) Catalan algebra,
- Find a polytope encoding clearly the algebra on parking functions, ...

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