Ky Fan Combinatorial Theorem and applications

Frédéric Meunier

June 30th, 2015

CERMICS, Optimisation et Systèmes

Outline

Ky Fan's combinatorial theorem and three applications:

1. Covering of the sphere.
2. Coloring of Kneser graphs.
3. Splitting necklaces.

Combinatorial Ky Fan's theorem

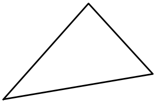
Figure: Ky Fan, 1914–2010

Simplex

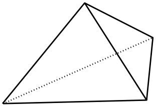
A simplex is the convex hull of affinely independent points.

Point

Edge



Triangle



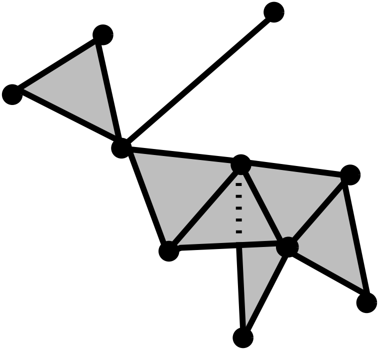
Tetrahedron

Etc.

Simplicial complex

K is a simplicial complex if it is a collection of simplices such that

- if τ is a face of $\sigma \in$ K, then $\tau \in$ K.
- the intersection of any two simplices is either empty or a face of both.



Alternating simplices

Let K be a simplicial complex and let $\lambda : V(\mathrm{K}) \to \{\pm 1, \pm 2, \ldots, \pm m\}$.

d-simplex σ is positively alternating if

$\lambda(V(\sigma))$ of the form $\{j_0, -j_1, \ldots, (-1)^d j_d\}$ with $1 \leq j_0 < j_1 < \cdots < j_d$

Combinatorial Ky Fan's theorem

Theorem

Let T *be a triangulation of the d-sphere S^d that is centrally symmetric. Let $\lambda : V(\mathrm{T}) \to \{\pm 1, \pm 2, \ldots, \pm m\}$ be a labeling such that*

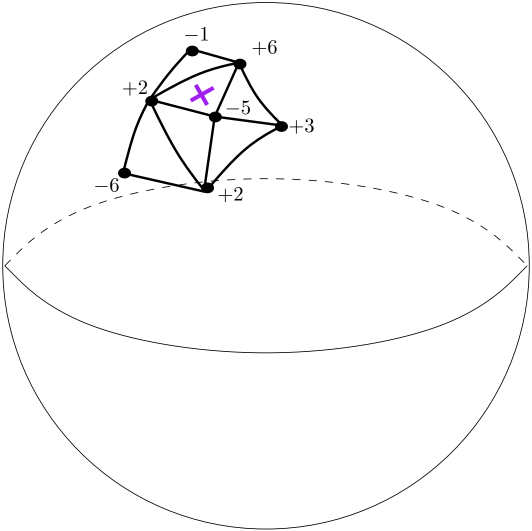
- $\lambda(-v) = -\lambda(v)$ *for all* $v \in V(\mathrm{T})$
- *There are no edges uv of* T *such that* $\lambda(u) + \lambda(v) = 0$.

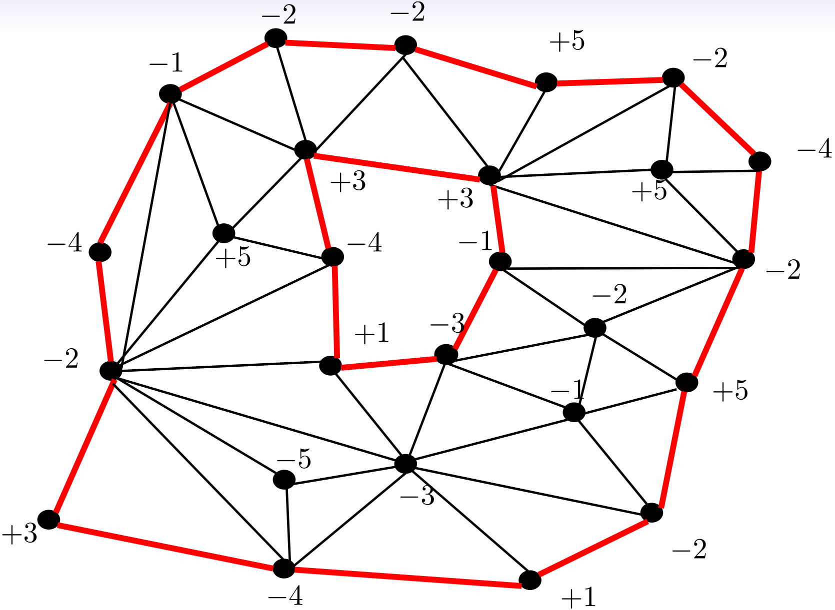
Then there is at least one positively alternating d-simplex.

d-simplex is positively alternating if

$$\lambda(V(\sigma)) \text{ of the form } \{j_0, -j_1, \ldots, (-1)^d j_d\} \text{ with } 1 \leq j_0 < j_1 < \cdots < j_d$$

Combinatorial Ky Fan's theorem



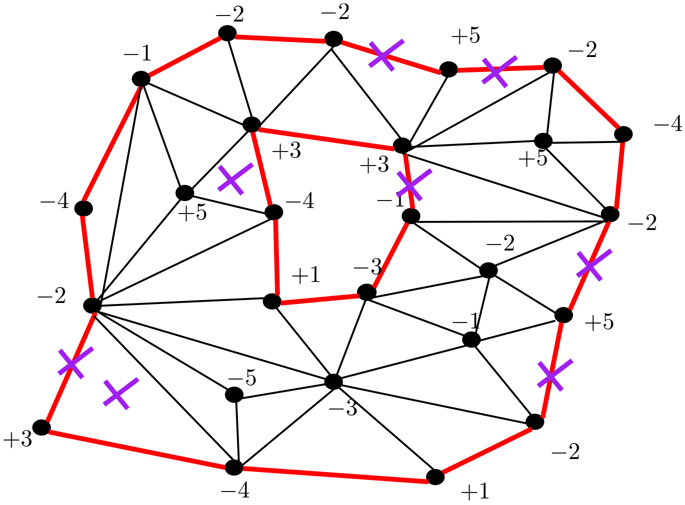

−2
−2
−1
+5
−2
−4
+3
+3
+5
−4
+5
−4
−1
−2
−2
+1
−3
−2
−1
+5
−5
−3
+3
−2
−4
+1

Combinatorial Stokes formula

$\beta^-(\mathrm{K})$: # negatively alternating triangles

$\beta^+(\mathrm{K})$: # positively alternating triangles

$\beta^-(\partial\mathrm{K})$: # negatively alternating edges on the boundary



$$\beta^-(\mathrm{K}) + \beta^+(\mathrm{K}) = \beta^-(\partial\mathrm{K}) \quad \text{mod } 2.$$

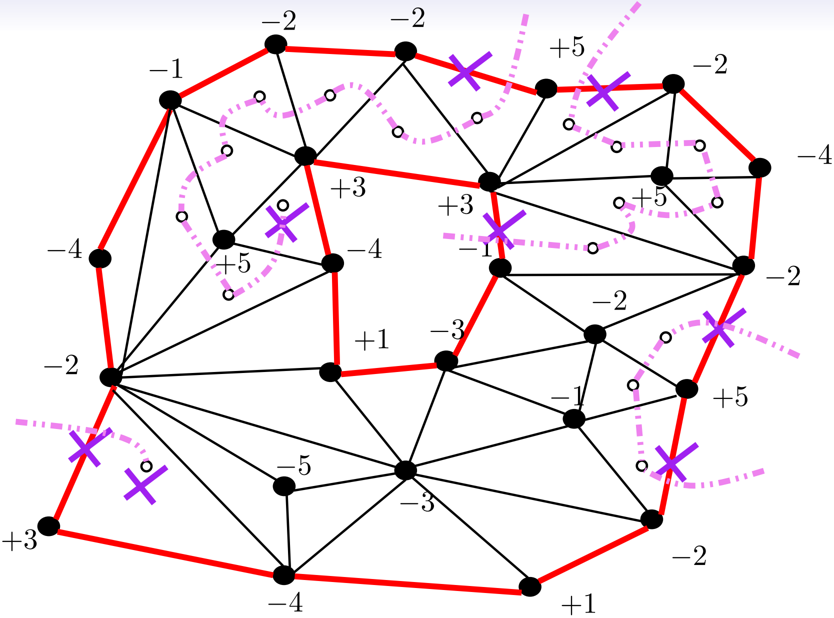
Combinatorial Stokes formula

K pseudomanifold of dimension d.

Let $\lambda : V(\mathrm{K}) \to \{\pm 1, \pm 2, \dots, \pm m\}$ be s.t. there are no edges uv of K with $\lambda(u) + \lambda(v) = 0$.

$\beta^-(\mathrm{K})$: number of negatively alternating d-simplices
$\beta^+(\mathrm{K})$: number of positively alternating d-simplices
$\beta^-(\partial\mathrm{K})$: number of negatively alternating $(d-1)$-simplices on the boundary

$$\beta^-(\mathrm{K}) + \beta^+(\mathrm{K}) = \beta^-(\partial\mathrm{K}) \qquad \text{mod } 2$$



−2
−2
+5
−2
−1
−4
+3
+3
+5
−4
−4
+5
−1
−2
−2
+1
−3
−2
−1
+5
−5
−3
+3
−2
−4
+1

Combinatorial Ky Fan's theorem

Theorem

Let T *be a triangulation of the* d*-sphere* S^d *that is centrally symmetric. Let* $\lambda : V(\mathsf{T}) \to \{\pm 1, \pm 2, \ldots, \pm m\}$ *be a labeling such that*

- $\lambda(-v) = -\lambda(v)$ *for all* $v \in V(\mathsf{T})$
- *There are no edges* uv *of* T *such that* $\lambda(u) + \lambda(v) = 0$*.*

Then there is at least one positively alternating d*-simplex.*

Application in topology

Theorem

Let $A_1, \ldots, A_m$ be m closed subsets of S^d satisfying the following conditions:

- *None of them contain antipodal points.*
- $\bigcup_{i=1}^{m}(A_i \cup (-A_i)) = S^d$.

Then there exist $d+1$ integers $1 \leq j_0 < \cdots < j_d \leq m$ such that

$$A_{j_0} \cap (-A_{j_1}) \cap \cdots \cap ((-1)^d A_{j_d}) \neq \emptyset.$$

Generalization of the Borsuk-Ulam theorem.

If f is a continuous $S^d \to \mathbb{R}^d$ map, then there is $\boldsymbol{x} \in S^d$ such that $f(\boldsymbol{x}) = f(-\boldsymbol{x})$.

Tucker's lemma

Lemma

Let T *be a triangulation of the* d*-sphere* S^d *that is centrally symmetric. Let* $\lambda : V(\mathrm{T}) \to \{\pm 1, \pm 2, \ldots, \pm m\}$ *be a labeling such that*

- $\lambda(-v) = -\lambda(v)$ *for all* $v \in V(\mathrm{T})$
- *There are no edges* uv *of* T *such that* $\lambda(u) + \lambda(v) = 0$.

Then $m \geq d + 1$.

Octahedral Ky Fan lemma

Lemma

Let $\lambda : \{+,-,0\}^n \setminus \{\mathbf{0}\} \to \{\pm 1, \dots, \pm m\}$ *s.t.*

- $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$ *for every* $\boldsymbol{x}$
- $\lambda(\boldsymbol{x}) + \lambda(\boldsymbol{y}) \neq 0$ *for every* $\boldsymbol{x} \preceq \boldsymbol{y}$

Then there is at least one positively alternating n-chain.

Positively alternating n-chain: $\boldsymbol{x}^1 \preceq \cdots \preceq \boldsymbol{x}^n$ with

$$\lambda(\{\boldsymbol{x}^1, \dots, \boldsymbol{x}^n\}) = \{j_1, -j_2, \dots, (-1)^{n-1} j_n\} \quad \text{and} \quad 1 \leq j_1 < j_2 < \cdots < j_n.$$

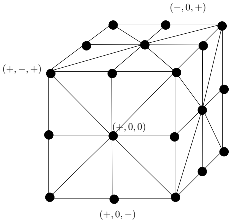
$$\boldsymbol{x} = (x_1, \dots, x_n) \preceq \boldsymbol{y} = (y_1, \dots, y_n) \quad \text{if} \quad x_i \neq 0 \Rightarrow y_i = x_i$$

Proof

⋆ $\{+,-,0\}^n \setminus \{\mathbf{0}\}$ is in one-to-one correspondence with the vertices of $\mathrm{sd}(\partial\square^n)$.

⋆ Chains correspond to simplices.

⋆ Apply the combinatorial Ky Fan's theorem.



Octahedral Tucker lemma

Lemma

Let $\lambda : \{+, -, 0\}^n \setminus \{\mathbf{0}\} \to \{\pm 1, \ldots, \pm m\}$ *s.t.*

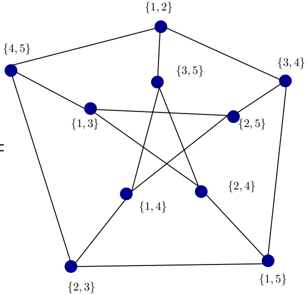
- $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$ *for every* $\boldsymbol{x}$
- $\lambda(\boldsymbol{x}) + \lambda(\boldsymbol{y}) \neq 0$ *for every* $\boldsymbol{x} \preceq \boldsymbol{y}$

Then $m \geq n$.

$$\boldsymbol{x} = (x_1, \ldots, x_m) \preceq \boldsymbol{y} = (y_1, \ldots, y_m) \quad \text{if} \quad x_i \neq 0 \Rightarrow y_i = x_i$$

Application: Combinatorial proof of the Lovász-Kneser theorem

Kneser graphs

$\mathrm{KG}(5,2) =$ 

n, k two integers s.t. $n \geq 2k$.

Kneser graph $\mathrm{KG}(n,k)$:

$$V(\mathrm{KG}(n,k)) = \binom{[n]}{k}$$

$$E(\mathrm{KG}(n,k))) = \left\{ AB : A, B \in \binom{[n]}{k},\ A \cap B = \emptyset \right\}$$

Lovász-Kneser theorem

Theorem
$\chi(\mathrm{KG}(n,k)) = n - 2k + 2.$

Original proof by Lovász in 1979, using algebraic topology.

$\chi(\mathrm{KG}(n,k)) \leq n - 2k + 2$ (easy: explicit coloring).

Matoušek proposed in 2003 a combinatorial (yet still topological) proof.

Matoušek's proof

- ⋆ $c : \binom{[n]}{k} \to [t]$ proper coloring of $\mathrm{KG}(n,k)$ with t colors.
- ⋆ Extension for any $U \subseteq [n]$: $c(U) = \max\{c(A) :\ A \subseteq U,\ |A| = k\}$.
- ⋆ $\boldsymbol{x}^+ = \{i :\ x_i = +\}$ and $\boldsymbol{x}^- = \{i :\ x_i = -\}$
- ⋆ $$\lambda(\boldsymbol{x}) = \begin{cases} |\boldsymbol{x}| & \text{if } |\boldsymbol{x}| \leq 2k-2,\ \min(\boldsymbol{x}^+) < \min(\boldsymbol{x}^-) \\ -|\boldsymbol{x}| & \text{if } |\boldsymbol{x}| \leq 2k-2,\ \min(\boldsymbol{x}^-) < \min(\boldsymbol{x}^+) \\ & \\ c(\boldsymbol{x}^+) + 2k - 2 & \text{if } |\boldsymbol{x}| \geq 2k-1,\ c(\boldsymbol{x}^+) > c(\boldsymbol{x}^-) \\ -c(\boldsymbol{x}^-) - 2k + 2 & \text{if } |\boldsymbol{x}| \geq 2k-1,\ c(\boldsymbol{x}^-) > c(\boldsymbol{x}^+) \end{cases}$$

Use the octahedral Tucker lemma

Apply the following lemma with $m = t + 2k - 2$.

Lemma

Let $\lambda : \{+, -, 0\}^n \setminus \{\mathbf{0}\} \to \{\pm 1, \ldots, \pm m\}$ *s.t.*

- $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$ *for every* $\boldsymbol{x}$
- $\lambda(\boldsymbol{x}) + \lambda(\boldsymbol{y}) \neq 0$ *for every* $\boldsymbol{x} \preceq \boldsymbol{y}$

Then $m \geq n$.

We have thus $t \geq n - 2k + 2$, as required.

Zig-zag theorem

Replace Tucker by Ky Fan (existence of the alternating chain), and get more.

Let $K_{q,q}$ denote the complete bipartite graph with q vertices on each side.

Theorem (Simonyi-Tardos 2006)

Suppose $\mathrm{KG}(n,k)$ *be colored properly with t colors. Then it contains a colorful copy of* $K_{\lfloor \frac{n-2k+2}{2} \rfloor, \lceil \frac{n-2k+2}{2} \rceil}$ *such that the colors alternate on both side.*

Let $K^*_{q,q} = K_{q,q} \setminus M$, where M is a perfect matching.

Theorem (Chen 2010)

Suppose $\mathrm{KG}(n,k)$ *be colored properly with $n-2k+2$ colors. Then it contains a colorful copy of* $K^*_{n-2k+2,n-2k+2}$.

Homomorphism of Kneser graphs

Let G and H be two graphs.
$f : V(G) \to V(H)$ is a graph homomorphism if $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$.

Conjecture (Stahl 1976)

There exists a graph homomorphism $\mathrm{KG}(n,k) \to \mathrm{KG}(n',k')$ if and only if $n' \geq qn - 2\ell$, where $k' = qk - \ell$.

Existence of a graph homomorphism $\mathrm{KG}(n,k) \to \mathrm{KG}(n-2,k-1)$: proved by Stahl in 1976. Case $n = 2k+1$ and $n' = 2k'+1$: also proved by Stahl in 1996.

Generalization: Kneser hypergraphs

n, k, r three integers s.t. $n \geq rk$.

Kneser hypergraph $\mathrm{KG}^r(n,k)$:

$$V(\mathrm{KG}^r(n,k)) = \binom{[n]}{k}$$

$$E(\mathrm{KG}^r(m,k)) = \left\{ \{A_1, \ldots, A_r\} : A_i \in \binom{[n]}{k},\ A_i \cap A_j = \emptyset \text{ for } i \neq j \right\}$$

Chromatic number

Theorem (Alon-Frankl-Lovász theorem)

$$\chi(\mathrm{KG}^r(m,\ell)) = \left\lceil \frac{m - r(\ell - 1)}{r - 1} \right\rceil$$

All proofs:

- if true for r_1 and r_2, then true for $r_1 r_2$.
- true when r is prime.

Original proof for the case r prime: similar as for Lovász-Kneser theorem, with deepest algebraic topology .

A combinatorial proof

Ziegler (2003) proposed a combinatorial proof via a Z_p-Tucker's lemma.

Assume p prime and $\mathrm{KG}^p(m,\ell)$ properly colored with t colors.

$Z_p = p$th roots of unity

With the help of coloring, build a map

$$\begin{array}{rccc} \lambda: & (Z_p \cup \{0\})^m \setminus \{\mathbf{0}\} & \longrightarrow & Z_p \times [t + p\ell - 2] \\ & \boldsymbol{x} & \longmapsto & (\underbrace{s(\boldsymbol{x})}_{\text{sign}}, \underbrace{v(\boldsymbol{x})}_{\text{absolute value}}) \end{array}$$

satisfying condition of a "Z_p-Tucker" lemma

- $\lambda(\omega\boldsymbol{x}) = \omega\lambda(\boldsymbol{x})$ for $\omega \in Z_p$
- condition on $\{\lambda(\boldsymbol{x}^1), \ldots, \lambda(\boldsymbol{x}^p)\}$ when $\boldsymbol{x}^1 \preceq \cdots \preceq \boldsymbol{x}^p$.

Second point satisfied by coloring condition: no p adjacent vertices get the same color.

Thus, $(p-1)(t-1) + p\ell - 1 \geq m$, i.e.

$$\boxed{t \geq \frac{m - p(\ell - 1)}{p - 1}}$$

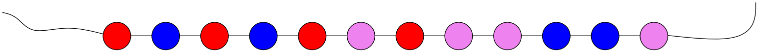
Application: the splitting necklace theorem

Two thieves and a necklace

n beads, t types of beads, a_i (even) beads of each type.

Two thieves: Alice and Bob.

Beads fixed on the string.

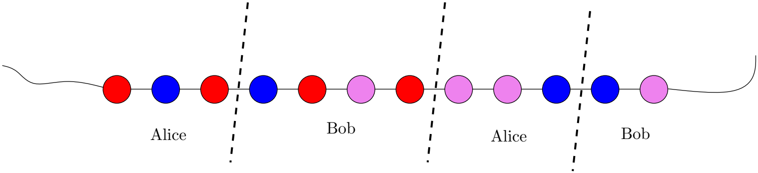


Fair splitting = each thief gets $a_i/2$ beads of type i

The splitting necklace theorem

Theorem (Alon, Goldberg, West, 1985-1986)

There is a fair splitting of the necklace with at most t cuts.



t is tight

t cuts are sometimes necessary:

Pálvölgyi's proof

$\star$ define alt($\boldsymbol{x}$) to be the number of sign changes when reading $\boldsymbol{x} \in \{+, -, 0\}^n$ from left to right (0 doesn't count).

$\star$ define $h(\boldsymbol{x})$ to be $\max\{\text{alt}(\boldsymbol{y}) : \boldsymbol{y} \succeq \boldsymbol{x}\}$.

$\star$ define $s(\boldsymbol{x})$ to be the first component of $\boldsymbol{y}$ realizing the maximum (well-defined!).

$$\star\ \lambda(\boldsymbol{x}) = \begin{cases} s(\boldsymbol{x})h(\boldsymbol{x}) & \text{if } h(\boldsymbol{x}) > t \\ \\ +i & \text{if } h(\boldsymbol{x}) \leq t \text{ and Alice gets } > a_i/2 \text{ beads of type } i \\ -i & \text{if } h(\boldsymbol{x}) \leq t \text{ and Bob gets } > a_i/2 \text{ beads of type } i \\ \\ & \text{and choose the smallest such } i \end{cases}$$

Use the octahedral Tucker lemma

Apply the following lemma with $m = n - 1$ (maximum possible number of sign changes in a $\boldsymbol{y}$).

Lemma

Let $\lambda : \{+, -, 0\}^n \setminus \{\mathbf{0}\} \to \{\pm 1, \ldots, \pm m\}$ *s.t.*

- $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$ *for every* $\boldsymbol{x}$
- $\lambda(\boldsymbol{x}) + \lambda(\boldsymbol{y}) \neq 0$ *for every* $\boldsymbol{x} \preceq \boldsymbol{y}$

Then $m \geq n$.

Contradiction. Such a λ doesn't exist.

$\Longrightarrow$ Existence of $\boldsymbol{x}$ with $h(\boldsymbol{x}) \leq t$ s.t. both Alice and Bob get $\leq a_i/2$ beads of type i, $\forall i$.

$\Longrightarrow$ Existence of $\boldsymbol{y} \succeq \boldsymbol{x}$ providing a fair splitting.

Open questions

⋆ Is there an elementary proof of the splitting necklace theorem?

⋆ What is the complexity of computing a fair splitting?

Generalization

q thieves.

Fair splitting = each thief gets a_i/q beads of type i

Theorem (Alon 1987)

There is a fair splitting of the necklace with at most $(q-1)t$ cuts.

Open questions

⋆ Is there a combinatorial proof using the Z_p-Tucker lemma?

⋆ Is there an elementary proof of the splitting necklace theorem?

⋆ What is the complexity of computing a fair splitting?