Colouring graphs

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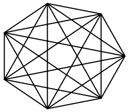
Theorem (Appel-Haken)

Every planar graph is 4-colourable.

Examples

- K_n Complete Graph (Clique) on n vertices :



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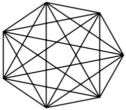

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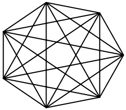

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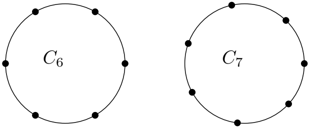

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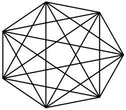

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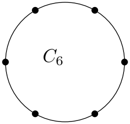

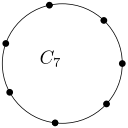

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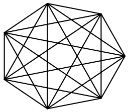

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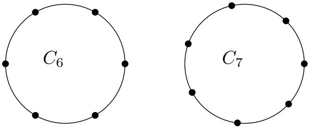

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Theorem (folklore)

A graph is bipartite (i.e. has chromatic number at most 2) if and only if it does not contain any odd cycle as a subgraph

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$$\chi(G') \geqslant \frac{|V(G')|}{\alpha(G')} \geqslant \frac{n^{1/k}}{4\log n} \geqslant k \text{ (for large enough } n)$$

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Theorem (Erdős - 1962)

For every k, there exists $\varepsilon > 0$ such that for all sufficielntly large n, there exists a graph G on n vertices with

- $\chi(G) > k$
- $\chi(G|_S) \leqslant 3$ *for every set S of size at most $\varepsilon.n$ in G.*

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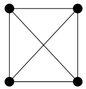
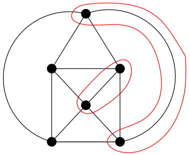


K_4



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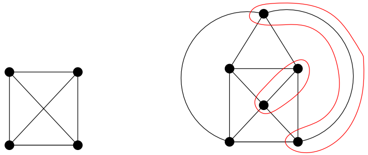

K_4 $\qquad$ *Octahedron*

Conjecture (Hadwiger - 1943)

$\chi(G) \geqslant k \Rightarrow G$ *contains* K_k *as a minor.*

(Proven for $k \leqslant 6$)

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(Weak perfect graph conjecture G perfect $\Rightarrow$ the complement of G is perfect. Proven by Lovász in 1972)

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Now our question is : what families $\mathcal{F}$ are chi-bounding?

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Conjecture (Gyarfas–Sumner)

If F is a forest, the class of graphs excluding F as an induced subgraph is chi-bounded.

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Little is really known :

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Scott proved the following very nice "topological" version of the conjecture

- For every tree T, the class of graphs excluding all subdivisions of T is chi-bounded.

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- The question originally came as a sub case of a more general question of Kalai and Meschulam.

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- Trinity changing paths : try to find vertices x and y such that many independent paths exist between the two.

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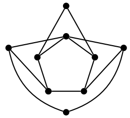

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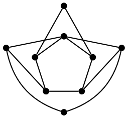

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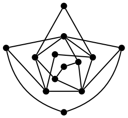

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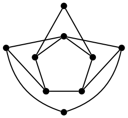

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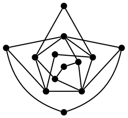

- If this other is present prove it.

$\mathcal{F}$ is an family of cycles.

Could the following conjecture be also true?

Conjecture

Every infinite family of cycles is chi-bounding.

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NO

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Conjecture

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NO
Using Erdős Theorem construct a sequence F_i such that

- $\chi(F_i) \geqslant i$
- $\text{girth}(F_i) > 2^{|F_{i-1}|}$.

Let $\mathcal{F}$ be the set of cycles that do NOT occur in any F_i.
Then $\mathcal{F}$ is not chi-bounding and is infinite (it contains at least all the $|F_i|$).
Even more it has upper density 1 since it contains every interval $[|F_i|, 2^{|F_i|}]$.

Conjecture (Scott-Seymour,2014)

If $I \subset \mathbb{N}$ *has bounded gaps (* $\exists k$ *s.t. every* k *consecutive integers contains an element of* F*), then* $\{C_i,\ i \in I\}$ *is* k*-bounding.*

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They proved (again very recently) that for any k, if G is triangle free and has sufficiently large chromatic number then it contain a sequence of holes of k consecutive lengths.
This contains our $0 \mod 3$ result, the long odd holes plus triangle.