Theory of oriented matroids and convexity

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A signed set X is a set $\underline{X}$ partitionned in two parts (X^+, X^-), where X^+ is the set of positive elements of X and X^- is the set of negatives elements.

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We say that X is a restriction of Y if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$. If A is a not signed set and X a signed set then $X \cap A$ designe the signed set Y with $Y^+ = X^+ \cap A$ et $Y^- = X^- \cap A$.

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Generally, given a signed set X and a set A we denote by $-_AX$ the signed set defined by $(-_AX)^+ = (X^+ \setminus A) \cup (X^- \cap A)$ and $(-_AX)^- = (X^- \setminus A) \cup (X^+ \cap A)$. We say that the signed set $-_AX$ is obtained by an reorientation of A.

A collection $\mathcal{C}$ of signed sets of a finite set E is the set of circuits of a oriented matroid on E if and only if the following axioms are verified :

$(C0)$ $\emptyset \notin \mathcal{C}$,

$(C1)$ $\mathcal{C} = -\mathcal{C}$,

$(C2)$ for any $X, Y \in \mathcal{C}$, if $\underline{X} \subseteq \underline{Y}$, then $X = Y$ or $X = -Y$,

$(C3)$ for any $X, Y \in \mathcal{C}, X \neq -Y$, and $e \in X^+ \cap Y^-$, there exists $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$.

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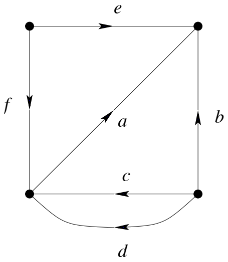
Notation. We may write $X = a\overline{bc}de$ the signed circuit X defined by $X^+ = \{a, d, e\}$ and $X^- = \{b, c\}$.

Oriented graph

Let G be an oriented graph. We obtain the signed circuits from the cycles of G.



Then,

$$\begin{aligned}\mathcal{C} \quad &= \{(a\bar{b}c),(a\bar{b}d),(a\bar{e}f),(c\bar{d}),(b\bar{c}\bar{e}f),(b\bar{d}\bar{e}f),\\ &\quad (\bar{a}b\bar{c}),(\bar{a}b\bar{d}),(\bar{a}e\bar{f}),(\bar{c}d),(\bar{b}ce\bar{f}),(\bar{b}de\bar{f})\}.\end{aligned}$$

Vector configuration

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We obtain an oriented matroid on E by considering the signed sets $X = (X^+, X^-)$ where

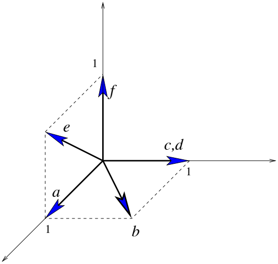
$$X^+ = \{i : \lambda_i > 0\} \text{ et } X^- = \{i : \lambda_i < 0\}$$

for all minimal dependencies among the $\mathbf{v}_i$.

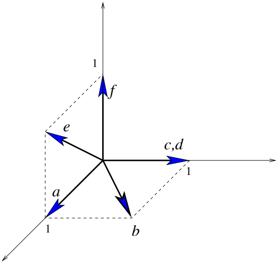
Let

$$A = \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

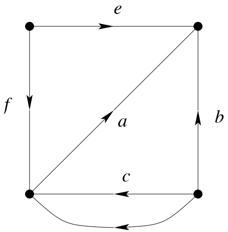
The columns of A correspond to the following vectors



We can check that the circuits of



are the same as those arising from



Configurations of points

Any configuration of points induce an oriented matroid in the affine space where the signed set of circuits are are the coefficients of minimal affine dependencies of the form

$$\sum_i \lambda_i \mathbf{v}_i = 0 \quad \text{with} \quad \sum_i \lambda_i = 0,\ \lambda_i \in \mathbb{R}$$

$$
\begin{array}{cc}
 & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
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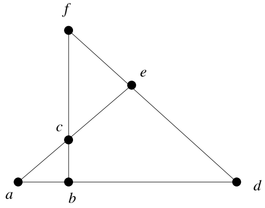

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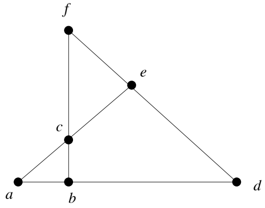

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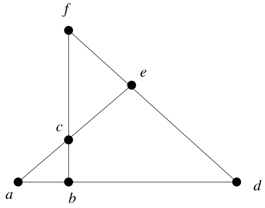

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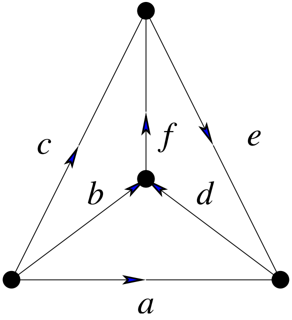
For instance, circuit $(a\bar{b}d)$ correspond to the affine dependecy $3(-1,0)^t - 4(0,0)^t + 1(3,0)^t = (0,0)^t$ with $3 - 4 + 1 = 0$.

The obtained oriented matroid is the one arising from



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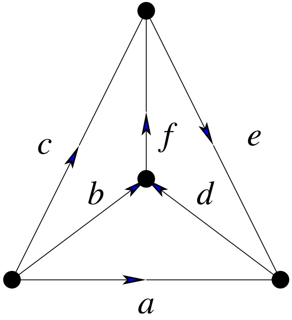

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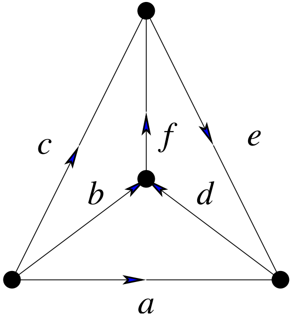

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For exemple, from circuit $(a\bar{b}d)$ we see that point b is in the segment $[a, b]$ and from circuit $(\bar{a}b\bar{e}f)$ the segment $[a, e]$ intersect the segment $[b, f]$

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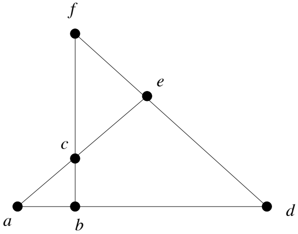
• $-_d M(\overline{A})$ is graphic. Moreover, it correspond to the oriented matroid



under the permutation
$\sigma(a) = b, \sigma(b) = a, \sigma(c) = c, \sigma(d) = d, \sigma(e) = f, \sigma(f) = e.$

Bases and Chirotope

$\mathcal{B}$ is the set of bases of an oriented matroid if and only if there is an application, called chirotope, $\chi : E^r \to \{+, -, 0\}$ such that.

(i) $\mathcal{B} \neq \emptyset$;

(ii) for any B and B' in $\mathcal{B}$ and $e \in B \setminus B'$ il there existes $f \in B' \setminus B$ such that $B \setminus e \cup f \in \mathcal{B}$;

(iii) $\{b_1, \ldots, b_r\} \in \mathcal{B}$ if and only if $\chi(b_1, \ldots, b_r) \neq 0$;

(iv) χ is alternating, i.e. $\chi(b_{\sigma(1)}, \ldots, b_{\sigma(r)}) = sign(\sigma)\chi(b_1, \ldots, b_r)$ for any $b_1, \ldots, b_r \in E$ and any permutation σ ;

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(v) (Three-terms Grassmann-Plücker relation) for any $b_1, \ldots, b_r, x, y \in E$, if $\chi(x, b_2, \ldots, b_r)\chi(b_1, y, b_3, \ldots, b_r) \geq 0$ and $\chi(y, b_2, \ldots, b_r)\chi(x, b_1, b_3, \ldots, b_r) \geq 0$ then $\chi(b_1, b_2, \ldots, b_r)\chi(x, y, b_3, \ldots, b_r) \geq 0$.

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Remark. In the realizable case, axiom (v) is directly verified with the Grassmann-Plücker's relation, it is thus a combinatorial reformulation :

$$det(b_1, \ldots, b_r) \cdot det(b'_1, \ldots, b'_r) = \sum_{1 \leq i \leq r} det(b'_i, b_2, \ldots, b_r) \cdot det(b'_1, \ldots, b'_{i-1}, b_1, b'_{i+1}, \ldots, b'_r).$$

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$$\chi(y, b_2, \ldots, b_r) = -C(e)C(f)\chi(x, b_2, \ldots, b_r)$$

where $\{x, b_2, \ldots, b_r\}$ and $\{y, b_2, \ldots, b_r\}$ are two bases with $x \neq y$ and $C(a)$ denote the sign of a in C, (one of the two opposite circuits contained in$\{x, y, b_2, \ldots, b_r\}$).

Arrangement of pseudospheres

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We have two connected components in $S^{d-1} \setminus S$, each homeomorphe to the d_1 dimensional ball (called sides of S).

A finite collection $\{S_1, \ldots, S_n\}$ of pseudo-spheres in S^{d-1} is an arrangement of pseudo-spheres if

$(PS1)$ for all $A \subseteq E = \{1, \ldots, n\}$ the set $S_A = \cap_{e \in A} S_e$ is a (topological) sphere

$(PS2)$ If $S_A \not\subseteq S_e$ for $A \subseteq E, e \in E$ and S_e^+, S_e^- denotes the two sides of S_e then $S_A \cap S_e$ is a pseudo-sphere of S_A having as sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

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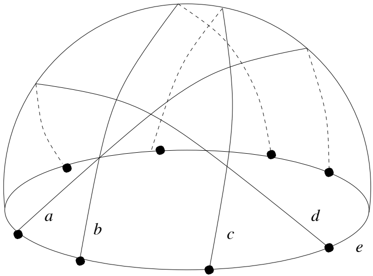
The arrangement is said to be essential if $S_E = \emptyset$.

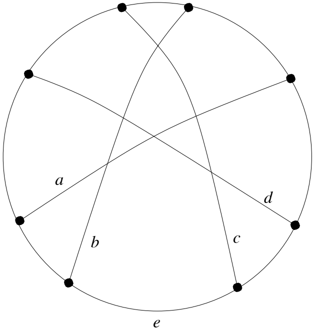
We say that the arrangement is signed if for each pseudosphere S_e, $e \in E$ it is chosen a positive and a negative side.

Topological representation

Topological Representation (Folkman+Lawrence) Any loop-free oriented matroid of rank $d+1$ (up to isomorphism) are in one-to-one correspondence with arrangements of pseudo-spheres in S^d (up to topological equivalence).

Arrangement of pseudolines





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An arrangement of pseudolines is simple if three or more pseudolines do not intersect in the same point.

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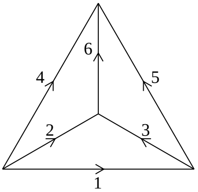
Theorem The set of acyclic orientations of M are in bijection with the set of cells of the corresponding arrangement of pseudospheres.


6
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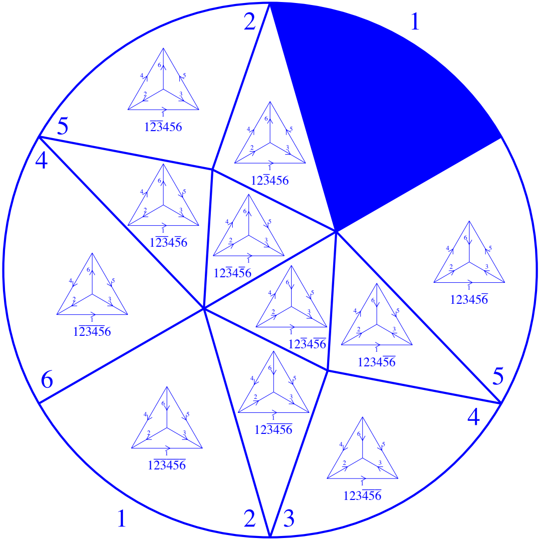
1
2
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4
6
5
4
1
2
3
$1\overline{23}456$
$12\overline{3}456$
$1\overline{23}4\overline{5}6$
$12\overline{3}4\overline{5}6$
$1\overline{2345}6$
$12345\overline{6}$
$12\overline{3}4\overline{56}$
$1234\overline{56}$
$12\overline{3456}$
$1\overline{23456}$
$123\overline{456}$

Theorem Let A_M be the arrangement of $H = \{h_1, \ldots, h_n\}$ pseudo-sphere corresponding to the oriented matroid M on n elements. Then, a cell of A_M that is bounded by $\{h_{i_1}, \ldots, h_{i_k}\}$ correspond to an acyclic reorientation of M having $[n] \setminus \{i_1, \ldots, i_k\}$ as interior points.

McMullen problem

A projective transformation $P : \mathbb{R}^d \to \mathbb{R}^d$ is such that $p(x) = \frac{Ax+b}{\langle c,x\rangle+\delta}$ where A is a linear transformation of $\mathbb{R}^d$, $b, c \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$ such that at least one of $c \neq 0$ or $\delta \neq 0$.

P is said permissible for a set $X \subset \mathbb{R}^d$ iff for all $x \in X, \langle c, x\rangle + \delta \neq 0$.

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Problem 1 Determine the largest integer $f(d)$ such that given any n points in general position in $\mathbb{R}^d$ there is a permissible projective transformation mapping these points onto the vertices of a convex polytope

Gale transforms

Given $\mathbf{a} = (a_1, \ldots, a_n)$ points in $\mathbb{R}^d$, we first convert the a_i into $\bar{a}_i = (a_i, 1) \in \mathbb{R}^{d+1}$. We suppose that $\bar{a}_i$ are $d+1$ affinely independent.

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Let V be the vector space generated by the rows of $(d + 1 \times n)$ matrix A having $\bar{a}_i$ as ith column. V is a $(d+1)$-dimensional subspace of $\mathbb{R}^n$.

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Let V be the vector space generated by the rows of $(d+1 \times n)$ matrix A having $\bar{a}_i$ as ith column. V is a $(d+1)$-dimensional subspace of $\mathbb{R}^n$.

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$$V^{\perp} = \{v \in \mathbb{R}^n \mid \langle u, v \rangle = 0 \text{ for all } u \in V\}.$$

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We have $dim(V^{\perp}) = n - d - 1$. Choose some basis $(b_1, \ldots, b_{n-d-1})$ of $V^{\perp}$ and let B be the $(n-d-1) \times n$ matrix with b_j as the jth row.

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Finally, let $\bar{g}_i \in \mathbb{R}^{n-d-1}$ be the ith column of B. The sequence $\bar{\mathbf{g}} = (\bar{g}_1, \ldots, \bar{g}_n)$ is the Gale transform of $\bar{\mathbf{a}}$.

Oriented matroid interpretation

Theorem Let $E = \{e_1, \ldots, e_n\}$ be a set of n points in $\mathbb{R}^d$, and suppose $\bar{E} = \{\bar{e}_1, \ldots, \bar{e}_n\}$ is a Gale transform of E. Then, $Aff(E)^{\perp} = Lin(\bar{E})$.

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Problem 2 Determine the smallest number $\lambda(d)$ such that any set X of λ points lying in general position in $\mathbb{R}^d$ can be partitioned in two sets A, B such that $conv(A \setminus x) \cap conv(B \setminus x) \neq \emptyset$ for all $x \in X$.

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Remark By using Gale transforms it can be proved that Problem 1 and Problem 2 are equivalent.

$$\lambda(d-1) = \min\{w : w \leq f(w-d-2)\}$$

$$f(d) = \max\{w : w \geq \lambda(w-d-2)\}$$

Back to McMullen problem

Problem 1 Determine the largest integer $f(d)$ such that given any n points in general position in $\mathbb{R}^d$ there is a permissible projective transformation mapping these points onto the vertices of a convex polytope.

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(Larman 1972) $2d + 1 \leq f(d) \leq (d+1)^2,\ f(d) = 2d + 1$ for $d = 2, 3$ and conjectured that $f(d) = 2d + 1$ for any $d \geq 2$.

Theorem (Las Vergnas 1985) $f(d) \leq d(d+1)/2$ for any $d \geq 2$.

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Remark Conjecture can easily be checked when $d = 2$ via the topological version.

Theorem (R.A. 2001) $f(d) \leq 2d + \lceil \frac{d}{2} \rceil$ for any $d \geq 2$.

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By using oriented matroid version version and Lawrence oriented matroids.

Lawrence oriented matroid

A Lawrence oriented matroid $\mathcal{M}$ of rank r on the totally ordered set $E = \{1, \ldots, n\}$, $r \leq n$, is a uniform oriented matroid obtained as the union of r uniform oriented matroids $\mathcal{M}_1, \ldots, \mathcal{M}_r$ of rank 1 on $(E, <)$.

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The chirotope χ corresponds to some Lawrence oriented matroid $\mathcal{M}_A$ if and only if there exists a matrix $A = (a_{i,j})$, $1 \leq i \leq r$, $1 \leq j \leq n$ with entries from $\{+1, -1\}$ (where the ith row corresponds to the chirotope of the oriented matroid $\mathcal{M}_i$) such that

$$\chi(B) = \prod_{i=1}^{r} a_{i,j_i}$$

where B is an ordered r-tuple $j_1 \leq \ldots \leq j_r$ elements of E.

Remarks

(*i*) The coefficients $a_{i,j}$ with $i > j$ or $j - n > i - r$ do not play any role in the definition of $\mathcal{M}_A$ (since they never appear in the chirotope). So, we may give them any arbitrary value from $\{+1, -1\}$ or ignore them completely.

(*ii*) An opposite chirotope $-\chi$ is obtained by reversing the sign of all the coefficients of a line of A.

(*iii*) The oriented matroid ${}_{\bar{c}}\mathcal{M}_A$ is obtained by reversing the sign of all the coefficients of a column c in A.

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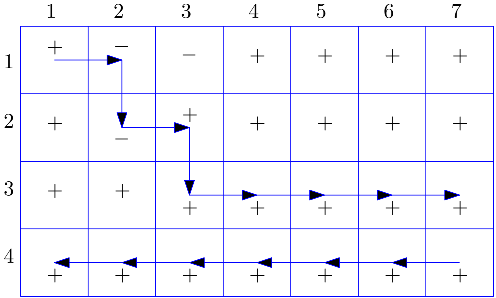

Lemma Let $\mathcal{M}_A$ be a Lawrence oriented matroid and A the matrix associated $A = (a_{i,j})$ with $1 \leq i \leq r$, $1 \leq j \leq n$ and entries from $\{+1, -1\}$. Then the following conditions are equivalent.

(a) $\mathcal{M}_A$ is cyclic,

(b) TT ends at $a_{r,s}$ for some $1 \leq s < n$,

(c) BT ends at $a_{1,s'}$ for some $1 < s \leq n$.

We say that TT and BT are parallel at column k with $2 \leq k \leq n-1$ in A if $TT = (a_{1,1}, \ldots, a_{i,k-1}, a_{i,k}, a_{i,k+1}, \ldots)$ and either $BT = (a_{r,n}, \ldots, a_{i,k+1}, a_{i,k}, a_{i,k-1}, \ldots)$ or $BT = (a_{r,n}, \ldots, a_{i+1,k+1}, a_{i+1,k}, a_{i+1,k-1}, \ldots)$, $1 \leq i \leq r$.

Lemma Let $\mathcal{M}_A$ be a Lawrence oriented matroid and A the matrix associated $A = (a_{i,j})$ with $1 \leq i \leq r$, $1 \leq j \leq n$ and entries from $\{+1, -1\}$. Then k is an interior element of $\mathcal{M}_A$ if and only if

(a) $BT = (a_{r,n}, \ldots, a_{1,2}, a_{1,1})$ for $k = 1$,

(b) $TT = (a_{1,1}, \ldots, a_{r,n-1}, a_{r,n})$ for $k = n$,

(c) TT and BT are parallel at k for $2 \leq k \leq n-1$.

Example

	1	2	3	4	5	6	7
1	$+$	$-$	$-$	$+$	$+$	$+$	$+$
2	$+$	$-$	$+$	$+$	$+$	$+$	$+$
3	$+$	$+$	$+$	$+$	$+$	$+$	$+$
4	$+$	$+$	$+$	$+$	$+$	$+$	$+$

We notice that $M_{A'}$ is acyclic and that 4, 5 and 6 are interior elements.

Observation There is a bijection between the set of all plain travels of A and the set of all acyclic reorientations of $\mathcal{M}_A$:

associate to P the set of elements of $\mathcal{M}_A$ that should be reoriented to transform P to the Top Travel of the new matrix $A^P = (a^P_{i,j})$ (obtained by reversing the signs of all coefficients of the columns in A corresponding the reoriented elements).

Generalizing McMullen problem

A d-polytope is k-neighbourly if for $k \leq \lceil \frac{d}{2} \rceil$ fixed, every subset of at most k vertices of the vertex set of the polytope is a face of the polytope.

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Theorem (Garcia-Colin 2014 Let $2 \leq k \leq \lceil \frac{d}{2} \rceil$ and $v(d,k)$ be the largest integer such that any $v(d,k)$ points in general position in $\mathbb{R}^d$ can be mapped by a permissible projective transformation onto points onto the vertices of a k-neighbourly convex polytope. Then, $d + \lceil \frac{d}{k} \rceil + 1 \leq v(d,k) < 2d - k + 1$.

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Proof of the upper bound (idea) Find a realizable, acyclic oriented matroid such that one of their acyclic reorientations contains at least on circuit C with $|C^+| \leq k$ (or $|C^-| \leq k$). Such a matroid couldn't possibly have a realization which is is a $k$neighbourly polytope.

Theorem (Garcia-Colin) Let $\lambda(d,k)$ be the smallest number such that for any set X of λ points lying in general position in $\mathbb{R}^d$ there exists a partition of X into two sets A, B such that $conv(A \setminus Y) \cap conv(B \setminus Y) \neq \emptyset$ for all $2 \leq k \leq \lceil \frac{d}{2} \rceil Y \subset X$, with $|Y| = k$. Then, $2d + k + 1 \leq \lambda(d,k) \leq (k+1)d + (k+2)$.

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Question Determine tha smallest $\lambda(d,s,k)$ number such that for any set X of λ points lying in general position in $\mathbb{R}^d$ there exists a partition of X into s sets $A_1, \ldots, A_s$ such that $\cap_{i=1}^{s} conv(A_i \setminus Y) \neq \emptyset$ for all $Y \subset X$, with $|Y| = k$.